MATH 20D Spring 2023 Lecture 2.

Vocabularly, Initial Value Problems, and Implicit Solutions

Announcements

- Homework 1 is posted, due 10pm next Tuesday via Gradescope.

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- MATLAB Gradescope has been created. Contact the head MATLAB TA Itai Maimon (`imaimon@ucsd.edu`) is you require access.

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  (a) You can impress your friends with cool math words.
  (b) We have the linguistic tools neccessary to isolate some nice classes of differential equations e.g. homogeous linear ODE's.

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The **degree** or **order** of an ODE is the largest value of k such that $\frac{d^ky}{dt^k}$ appears in the equation.

Example: PDE's vs ODE's

Determine which equations are PDE's and which are ODE's

- (Hermite's Equation for a quantum harmonic osciallator)

$$\frac{d^2y}{dt^2} - 2x\frac{dy}{dt} - \lambda y = 0$$

where λ is a constant.

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Linear and Nonlinear ODE's

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The true **star** of this class with be the **the linear ODE** of order $\leqslant 2$.

Verifying Differential Equations I

- There no known method for solving a general **linear** ODE

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Example

Verify that

$$\phi\colon \mathbb{R} - \{0\} \to \mathbb{R}, \qquad \phi(x) = x^2 - x^{-1}$$

is a solution to the **initial value problem**

$$\frac{d^2\phi}{dx^2} - \frac{2}{x^2}\phi = 0, \qquad \phi(1) = 0, \quad \phi'(1) = 3.$$

Initial Value Problems

Definition

An **n-th order linear initial value problem (IVP)** is an n-th order linear ODE

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

together with a family of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

such that $x_0, y_0, y_1, \ldots, y_{n-1}$ are fixed constants and the functions

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Theorem

An n-th order linear IVP has a unique solution $y_{\text{sol}}(t)$ on any interval $I \subseteq \mathbb{R}$ on which the function $a_{n-1}(t), \ldots, a_0(t)$, and $g(t)$ are all continuous.

Example

Verify the statement of the previous theorem for the IVP

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with the initial condition $y(0) = 1$.

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Existence: Recall the familiar exponential function

$$y\colon \mathbb{R} \to \mathbb{R}_{>0}, \qquad y(x) = e^x.$$

One way to define e^x is by the Taylor series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

- The series converges absolutely for all $x \in \mathbb{R}$ (ratio test).
- To justify the formal calculation

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = e^x$$

requires material from MATH 140B.